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Oscillation Behavior of Arbitrary Order Neutral Differential Equations

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Abstract—In this paper, we shall consider arbitrary order neutral differential equation with forcing term. Sufficient conditions for the oscillation behavior of solutions for this differential equation are presented. We present new results and also extend some of the known results. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Neutral differential equations, Oscillations, Nonlinear equations.

1. INTRODUCTION

In this paper, we study the n^{th}-order neutral differential equations

$$\left[x(t)|x(t)|^{\alpha-1}+\sum_{i=1}^{m}p_i(t)x(\tau_i(t))\right]^{(n)}+\sum_{j=1}^{2}\lambda_j q_j(t)f_j(x(\sigma_j(t)))=h(t), \tag{1}$$

where $\lambda_1,\lambda_2\in\{-1,0,1\}$; $\alpha\ge 1$, $p_i,\tau_i,q_j,\sigma_j\in C([t_0,\infty),\mathbb{R})$ for $i=1,2,\dots,m$ and $j=1,2$, and $f_j,h\in C(\mathbb{R},\mathbb{R})$, $j=1,2$; $p_i(t)\ge 0$, $\tau_i(t)\le t$ and $\tau_i(t)\to\infty$ as $t\to\infty$ for $i=1,2,\dots,m$ and $\sum_{i=1}^{m}p_i(t)<1$; $q_j(t)>t^{-n}$ as $t\to\infty$ for $j=1,2$; $\sigma_1(t)>t$, $\sigma_2(t)<t$, and $\sigma_2(t)\to\infty$ as $t\to\infty$, and $\sigma_1(t)$ and $\sigma_2(t)$ are nondecreasing; $xf_j(x)\ge d_jx^2$ for some $d_j>0$ for $j=1,2$.

We develop certain theorems related to the oscillatory behavior and provide sufficient conditions for the above equation to be oscillatory. The oscillatory behavior of neutral differential equation of n^{th} order has been the subject of several papers [1–5].

In this paper, we improve and extend some of these results to n^{th}-order nonlinear neutral differential equations. A solution $x(t)\in C([t_0,\infty),\mathbb{R})$ of (1) is called oscillatory if $x(t)$ has arbitrarily large zeros in $[t_0,\infty)$, $t_0>0$. Otherwise, $x(t)$ is called nonoscillatory.

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2. MAIN THEOREMS

THEOREM 1. *Let $\lambda_2 = 0$ and $(-1)^n\lambda_1 = -1$. Assume that there exists an oscillatory function $g(t)$ such that*

$$g^{(n)}(t) = h(t), \qquad \lim_{t\to\infty} g(t) = \lim_{t\to\infty} g'(t) = 0, \tag{2}$$

and that every solution of second-order ordinary differential equation

$$u''(t) + \frac{\mu d_1}{(n-1)!}(t-T)^{n-l-1+(l-1)/\alpha}\phi_1(t)|u(t)|^{1/\alpha}\operatorname{sgn}(u(t)) = 0, \tag{3}$$

where $\phi_1(t) = [1 - \sum_{i=1}^m p_i(\sigma_1(t))]^{1/\alpha} q_1(t)$ and $l \in \{1, 2, \dots, n-1\}$, is oscillatory for some constant μ, $0 < \mu < 1$ and for every $T > 0$.

(i) *If n is odd, then every solution $x(t)$ of (1) is either oscillatory or satisfies*

$$\liminf_{t\to\infty} |x(t)| = 0.$$

(ii) *If n is even, then every solution $x(t)$ of (1) is either oscillatory or else*

$$\lim_{t\to\infty} |x(t)| = \infty \qquad \text{or} \qquad \liminf_{t\to\infty} |x(t)| = 0.$$

PROOF. Let $x(t)$ be a nonoscillatory solution of (1) and $z(t)$ be a function defined by

$$z(t) = x(t)|x(t)|^{\alpha-1} + \sum_{i=1}^m p_i(t)x(\tau_i(t)).$$

We may assume that $x(t)$ is eventually positive. The case where $x(t)$ is eventually negative can be treated similarly. One can see that $z(t)$ and $x(\sigma_1(t))$ are also eventually positive. Consider the function $y(t) = z(t) - g(t)$. Then, from (1),

$$\lambda_1 y^{(n)}(t) = -q_1(t) f_1(x(\sigma_1(t))), \tag{4}$$

so that $y^{(n)}(t)$ is eventually of one-signed. Thus, the lower derivatives $y^{(i)}(t)$, $0 \le i \le n-1$, are monotone and one-signed for all large $t \ge t_0$. If $y(t) < 0$ for $t \ge t_0$, then $0 < z(t) < g(t)$, $t \ge t_0$, which shows that $g(t)$ takes on only the positive values for arbitrarily large t. But this contradicts $g(t)$ being an oscillatory function, so we must have $y(t) > 0$ for $t \ge t_0$. By Kiguradze's [6] lemmas, there exists an integer $l \in \{0, 1, \dots, n\}$ with $(-1)^{n-l-1}\lambda_1 = 1$ such that

$$\begin{aligned} y^{(i)}(t) &> 0, &&\text{on } [T, \infty), \quad i = 0, 1, 2, \dots, l, \\ (-1)^{i-l} y^{(i)}(t) &> 0, &&\text{on } [T, \infty), \quad i = l, l+1, \dots, n, \end{aligned} \tag{5}$$

for some $T \ge t_0$. Suppose that $0 < l < n$. Then, by Taylor's theorem, we have

$$y^{(l)}(t) = \sum_{j=0}^{n-l-1} \frac{(-1)^j y^{(l+j)}(\tau)}{j!}(\tau - t)^j + \frac{(-1)^{n-l-1}\lambda_1}{(n-l-1)!}\int_t^\tau (s-t)^{n-l-1}\left(-\lambda_1 y^{(n)}(s)\right) ds.$$

Now using (5), we obtain

$$y^{(l)}(t) \ge \frac{1}{(n-l-1)!}\int_t^\tau (s-t)^{n-l-1}\left(-\lambda_1 y^{(n)}(s)\right) ds, \qquad T \le t \le \tau.$$

Letting $\tau \to \infty$ in the last equation and integrating from T to t, we have

$$\begin{aligned} y^{(l-1)}(t) &\geq y^{(l-1)}(T) + \frac{1}{(n-l-1)!}\int_T^t \int_s^\infty (r-s)^{n-l-1}\left(-\lambda_1 y^{(n)}(r)\right) dr\, ds \\ &= y^{(l-1)}(T) + \frac{1}{(n-l-1)!}\int_t^\infty \left[\int_T^t (r-s)^{n-l-1}\, ds\right]\left(-\lambda_1 y^{(n)}(r)\right) dr \\ &\quad + \frac{1}{(n-l-1)!}\int_T^t \left[\int_T^r (r-s)^{n-l-1}\, ds\right]\left(-\lambda_1 y^{(n)}(r)\right) dr, \qquad t \geq T. \end{aligned} \tag{6}$$

Now by making use of the inequality

$$\int_T^t (r-s)^{n-l-1}\, ds \geq \frac{1}{n-l}(t-T)(r-T)^{n-l-1}, \qquad T \leq t \leq r,$$

it follows from (6) that

$$\begin{aligned} y^{(l-1)}(t) \geq y^{(l-1)}(T) &+ \frac{1}{(n-l)!}\int_T^t (r-T)^{n-l}\left(-\lambda_1 y^{(n)}(r)\right) dr \\ &+ \frac{(t-T)}{(n-l)!}\int_t^\infty (r-T)^{n-l-1}\left(-\lambda_1 y^{(n)}(r)\right) dr, \qquad t \geq T. \end{aligned}$$

Let the right-hand side of the above inequality be $u(t)$. Then, we can show that $u(t)$ is positive and satisfies

$$u''(t) + \frac{1}{(n-l)!}(t-T)^{n-l-1}\left(-\lambda_1 y^{(n)}(t)\right) = 0, \qquad t \geq T. \tag{7}$$

Since $(-1)^n\lambda_1 = -1$ and $(-1)^{n-l-1}\lambda_1 = 1$, l is even. Therefore, y is eventually increasing and concave up when $0 < l < n$. Then, using the nature of $y(t)$ and (2) with $y(t) + g(t) = z(t)$, we see that $z(t)$ is increasing for large t as well. Thus,

$$z(t) = x(t)|x(t)|^{\alpha-1} + \sum_{i=1}^m p_i(t)x(\tau_i(t)) \leq [x(t)]^\alpha + \sum_{i=1}^m p_i(t)z(t)$$

or

$$\left(1 - \sum_{i=1}^m p_i(t)\right) z(t) \leq [x(t)]^\alpha.$$

Since $y(t)$ is positive, increasing, and $\lim_{t\to\infty} g(t) = 0$, we have

$$z(t) \geq \mu^\alpha y(t), \qquad t \geq T,$$

where μ is the same constant as in (3), for large enough T. Then,

$$[x(t)]^\alpha \geq \mu^\alpha \left(1 - \sum_{i=1}^m p_i(t)\right) y(t), \qquad t \geq T. \tag{8}$$

On the other hand, it can be shown that $y(t)$ satisfies

$$y(t) \geq \frac{1}{l!}(t-T)^{l-1} y^{(l-1)}(t), \qquad \text{for } t \geq T. \tag{9}$$

Combining inequalities (8) and (9) with the fact that $y^{(l-1)}(t) \geq u(t)$, we have

$$\begin{aligned} x(\sigma_1(t)) &\geq \left[\frac{1}{l!}(t-T)^{l-1}\mu^\alpha \left(1 - \sum_{i=1}^m p_i(\sigma_1(t))\right) u(t)\right]^{1/\alpha} \\ &\geq \frac{1}{l!}(t-T)^{(l-1)/\alpha}\mu \left[1 - \sum_{i=1}^m p_i(\sigma_1(t))\right]^{1/\alpha} u^{1/\alpha}(t), \qquad t \geq T. \end{aligned} \tag{10}$$

Multiply both sides of (10) by $q_1(t)d_1$ and use the fact $xf_1(x) > d_1x^2$. Then, we have from (7) that

$$u''(t) + \frac{\mu d_1}{(n-1)!}(t-T)^{n-l-1+(l-1)/\alpha}\phi_1(t)u^{1/\alpha}(t) \le 0, \qquad t \ge T. \tag{11}$$

When we apply the result of Atkinson [7] to (11), we see that equation (3) has an eventually positive solution, which is a contradiction.

It is clear that $l = n$ is only possible when n is even. In that case, $\lambda_1 = -1$ and

$$y^{(i)}(t) \to \infty, \qquad \text{as } t \to \infty, \quad \text{for } i = 0, 1, \dots, n-2,$$

which implies that $x(t) \to \infty$ as $t \to \infty$, since $[x(t)]^\alpha \ge (1 - \sum_{i=1}^m p_i(t))z(t)$, $z(t) = y(t) + g(t)$ and $\lim_{t\to\infty} g(t) = 0$.

If $l = 0$, $y(t)$ decreases to a nonnegative number, say c, as t goes to infinity. Since

$$\int^\infty t^{n-1}q_1(t)\,dt = \infty,$$

it follows that

$$\liminf_{t\to\infty} x(t) = 0. \qquad \blacksquare$$

COROLLARY 1. *Suppose that there exists an oscillatory function $g(t)$ satisfying (2). Then, the conclusion of Theorem 1 holds if*

$$\int_a^\infty s(s-T)^{n-l-1+(l-1)/\alpha}\phi_1(s)\,ds = \infty, \qquad l \in \{1, 2, \dots, n-1\}, \tag{12}$$

for every $T > 0$.

EXAMPLE 1. Consider the following functional differential equation:

$$\begin{aligned}\left[x(t)|x(t)|^2 + \left(1 - e^{-t/2}\right)e^{-\pi}x(t-\pi)\right]'' &- e^{t/2+2\pi}x(t+2\pi) = 18e^{-3t}\cos^2 t\sin t \\ &+ 6e^{-3t}\cos t + \frac{5}{4}e^{-3t/2}\cos t - 2e^{-t}\sin t + 3e^{-3t/2}\sin t - e^{-t/2}\cos t,\end{aligned}$$

so that $\lambda_1 = -1$, $\alpha = 3$, $p(t) = (1-e^{-t/2})e^{-\pi}$, $\tau(t) = t - \pi$, $\sigma_1(t) = t + 2\pi$, $q(t) = e^{t/2+2\pi}$, $g(t) = e^{-3t}\cos^3 t + (12/25)e^{-t/2}\cos t - e^{-t}\cos t + e^{-3t/2}\cos t + (16/25)e^{-t/2}\sin t$, and $f(x) = x$.

We can see that Corollary 1 is satisfied, and therefore, the conclusion of Theorem 1 holds. One can verify that $x(t) = e^{-t}\cos t$ is a solution of this equation.

THEOREM 2. *Let $\lambda_1 = 0$ and $(-1)^n\lambda_2 = -1$. Assume that there exists an oscillatory function $g(t)$ satisfying (2) and that every solution of second-order ordinary differential equation*

$$u''(t) + \frac{\delta d_2}{(n-1)!}(\sigma_2(t)-T)^{n-l-1+(l-1)/\alpha}\phi_2(t)|u(\sigma_2(t))|^{1/\alpha}\mathrm{sgn}(u(\sigma_2(t))) = 0, \tag{13}$$

where $\phi_2(t) = [1 - \sum_{i=1}^m p_i(\sigma_2(t))]^{1/\alpha}q_2(t)$ and $l \in \{1, 2, \dots, n-1\}$, is oscillatory for some constant δ, $0 < \delta < 1$ and for every $T > 0$. Then, the conclusion of Theorem 1 holds.

PROOF. Let $x(t)$ be a nonoscillatory solution of (1) and $z(t)$ be a function defined by

$$z(t) = x(t)|x(t)|^{\alpha-1} + \sum_{i=1}^m p_i(t)x(\tau_i(t)).$$

Proceeding as in the proof of Theorem 1 until we reach (10) with μ replaced by δ in (8) and combining inequalities (8) and (9) with the fact that $y^{(l-1)}(t) \geq u(t)$, we have

$$\begin{aligned} x(\sigma_2(t)) &\geq \left[\delta^\alpha \left(1 - \sum_{i=1}^{m} p_i(\sigma_2(t))\right) y(\sigma_2(t))\right]^{1/\alpha} \\ &\geq \frac{\delta}{l!}(\sigma_2(t) - T)^{(l-1)/\alpha} \left[1 - \sum_{i=1}^{m} p_i(\sigma_2(t))\right]^{1/\alpha} u^{1/\alpha}(\sigma_2(t)), \end{aligned} \tag{14}$$

for $t \geq T$. Multiply both sides of (14) by $q_2(t)d_2$ and use the fact $xf_2(x) > d_2x^2$. Then, we have from (7) that

$$u''(t) + \frac{\delta d_2}{(n-1)!}(\sigma_2(t) - T)^{n-l-1+(l-1)/\alpha}\phi_2(t)u^{1/\alpha}(\sigma_2(t)) \leq 0, \qquad t \geq T. \tag{15}$$

Applying now a result of Onose [8] to (15), we see that equation (13) is nonoscillatory, which is a contradiction.

Now suppose that $l = n$. It is clear that $l = n$ is only possible when n is even ($\lambda_2 = -1$). In that case,

$$y^{(i)}(t) \to \infty, \qquad \text{as } t \to \infty, \quad \text{for } i = 0, 1, \dots, n-2,$$

which implies that

$$x(t) \to \infty, \qquad \text{as } t \to \infty,$$

since $[x(t)]^\alpha \geq (1 - \sum_{i=1}^{m} p_i(t))z(t)$, $z(t) = y(t) + g(t)$, and $\lim_{t\to\infty} g(t) = 0$.

Now $l = 0$ is possible when $\lambda_2 = 1$ and n is odd, or $\lambda_2 = -1$ and n is even. In both cases, $y(t)$ decreases to a nonnegative number, say c, as t goes to infinity. Since

$$\int^\infty t^{n-1} q_2(t)\, dt = \infty,$$

we have

$$\liminf_{t\to\infty} x(t) = 0. \qquad \blacksquare$$

THEOREM 3. *Let $\lambda_1 = \lambda_2 = 1$. Assume that there exists an oscillatory function $g(t)$ satisfying (2). In addition, suppose that either equation (3) is oscillatory for some $0 < \mu < 1$ or equation (13) is oscillatory for some $0 < \delta < 1$ and for every $T > 0$. Then, if n is odd, every solution $x(t)$ of (1) is either oscillatory or satisfies*

$$\liminf_{t\to\infty} |x(t)| = 0.$$

PROOF. Let $x(t)$ be a nonoscillatory solution of (1). Let $x(t) > 0$ for some $t \geq t_0$. Proceeding as in the proof of Theorem 1, we get

$$u''(t) + \frac{1}{(n-l)!}(t - T)^{n-l-1}\left[q_1(t)f_1(x(\sigma_1(t))) + q_2(t)f_2(x(\sigma_2(t)))\right] = 0, \qquad t \geq T, \tag{16}$$

for $0 < l < n$. From the last equation, we deduce that

$$u''(t) + \frac{1}{(n-l)!}(t - T)^{n-l-1}q_1(t)f_1(x(\sigma_1(t))) \leq 0, \qquad t \geq T, \tag{17}$$

and

$$u''(t) + \frac{1}{(n-l)!}(t - T)^{n-l-1}q_2(t)f_2(x(\sigma_2(t))) \leq 0, \qquad t \geq T. \tag{18}$$

In view of Atkinson's [7] and Onose's [10] result, (17) and (18) have an eventually positive solution, respectively, but this is a contradiction to the oscillatory behavior of (3) and (13). The remaining part of the proof is similar to the last part of the proof of Theorem 1, so it is omitted. $\blacksquare$

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